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On the solution of damped wave conduction and relaxation equation in a semi-infinite medium subject to constant wall flux

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ABSTRACT

Eight reasons are given to seek a generalized Fourier's law of heat conduction and relaxation. Bounded solutions are obtained for the damped wave conduction and relaxation equation in one dimension in Cartesian coordinates for a semi-infinite medium subject to the constant wall flux boundary condition for the dimensionless heat flux and dimensionless temperature. Three different methods were employed. In the first approach the method of Laplace transforms was used. The solutions are domain restricted. Three regimes can be identified (a) zero transferring regime; (b) rising regime and (c) falling regime. In the second approach a generalized substitution is used to transform the hyperbolic PDE into a parabolic PDE. The transform selected is one with spatiotemporal symmetry. The resulting parabolic PDE can be solved for using the Boltzmann transformation. In the third approach the damping term was first removed from the governing equation. The resulting equation was transformed into a Bessel differential equation using a spatiotemporal symmetric transformation variable. A approximate solution for the flux was obtained. The inertial regime, rising and falling regimes were identified in the solution. A Chebyshev polynomial approximation was used for the integrand with modified Bessel composite function in space and time. Telescoping power series leads to more useful expression for transient heat flux. The temperature and heat flux solutions at the wave front were also developed. The solution for transient heat flux from the method of relativistic transformation is compared side by side with the solution for transient temperature from the method of Chebyshev economization. Both solutions are within 12% of each other. For conditions close to the wave front the solution from the Chebyshev economization is expected to be close to the exact solution and was found to be within 2% of the solution from the method of relativistic transformation. Far from the wave front, i.e., close to the surface the numerical error from the method of Chebyshev economization is expected to be significant and verified by a specific example. The solutions for dimensionless heat flux and dimensionless temperature is found to be continuous across the wave front without any singularities or jumps.

1. Introduction

The Fourier's law of heat conduction is not universal. There are eight reasons to seek a generalized Fourier's law of heat conduction. These are

(1) Contradiction of Fourier's law with the theory of microscopic reversibility of Onsager [1].
(2) Oscillatory discharge of heat in good thermal conductors at low temperature (Nernst [2]).
(3) Ultra fast laser heating of metals [3], delayed ignition of solid propellant [4], cannot be described using Fourier's parabolic equations.
(4) Landau and Lifshitz [5] noted that the speed of heat cannot be greater than the speed of light.
(5) Singularities can be seen in the solutions to the Fourier parabolic model for industrially important cases:
   (i) Blow-up [6] as time goes to zero of surface flux, during transient heat conduction in a semi-infinite medium subject to a CWT, constant wall temperature in Cartesian coordinates.
   (ii) Surface flux during transient heat conduction in a finite slab of width $2a$ subject to a step change in surface temperature.
   (iii) Temperature term in the CWF, constant wall flux problem in cylindrical coordinates in infinite medium solved for by using the Boltzmann transformation [7] leading to a solution in exponential integral [8–12].
   (iv) In the short time limit the parabolic equations are solved for by Boltzmann transformation for an infinite sphere and a singularity is found in the temperature.

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Nomenclature

C_p	heat capacity (J/kg/K)
erf	error function
g	function of X only
g'	first derivative of g wrt X
g''	second derivative of g wrt X
I_0	modified Bessel function of the zeroth order and first kind
I_1	modified Bessel function of the first order and first kind
J_0	Bessel function of the zeroth order and first kind
k	thermal conductivity (w/m/K)
K_0	modified Bessel function of the zeroth order and second kind
q	heat flux (w/m^2)
q^*	dimensionless heat flux (q/q_o)
q_0^*	dimensionless flux at surface $\tau_r^{1/2}\ q_0/(k\rho C_p)^{1/2}(T_0)$
q_o	constant wall flux (w/m^2)
T	temperature (K)
T_s	surface temperature (K)
T_0	initial temperature (K)
u	dimensionless temperature $(T - T_0)/(T_0)$
V	function of time only
V'	first derivative of V wrt τ
V''	second derivative of V wrt τ
W	wave flux ($u = W\exp(-\tau/2)$)
X	dimensionless ordinate $x/(\alpha\tau_r)^{1/2}$
Y_0	Bessel function of the zeroth order and second kind

Greek symbols

α	thermal diffusivity (m^2/s)
β	substitution constant
ρ	density (kg/m^3)
τ_r	relaxation time (s)
τ	dimensionless time (t/τ_r)
η	spatiotemporal transformation variable ($\tau^2 - X^2$)
θ	spatiotemporal transformation variable ($X \pm \tau(1-\beta)^{1/2}$)
ψ	Boltzmann transformation variable ($\theta/(4\beta\tau)^{1/2}$)

(6) Fourier's law was developed from empirical observations at steady state and when used in transient applications is a extrapolation and not adequately confirmed by careful experimentation.
(7) Over prediction of theory to experiment was found in important industrial systems such as fluidized bed heat transfer to immersed surfaces [13], CPU overheating [14], gel electrophoresis [15], restriction mapping [16] adsorption [17], nuclear fuel rod [18], drug delivery systems [19], when parabolic Fourier model is used indicating another mechanism that has not been considered well.
(8) The Casimir limit [20] or during transfer of heat in nanoscale regions the Fourier law is replaced with*Equation of phonon radiative transport*. In this limit the flux is described by an expression similar to the one used in radiation heat transfer [21,22]. The heat transport, for example, in dielectric crystalline materials is believed to be primarily by atomic or crystal vibrations. These vibrations travel as waves and the energy of the waves quantitated is the phonon [23].

Boley [24] found that the addition of the second derivative in time of temperature to the governing equation is the only way to remove the singularities found in the solution to parabolic heat conduction equations. The generalized Fourier's law of heat conduction can be written as

$$q = -k\frac{\partial T}{\partial x} - \tau_r\frac{\partial q}{\partial t}. \qquad (1)$$

This is the damped wave conduction and relaxation equation. When the relaxation time, τ_r is zero Eq. (1) will revert to the Fourier's law of heat conduction. Reference to the use of Eq. (1) can be traced back to Maxwell [25] and Morse and Feshbach [26] Cattaneo [27], [28] and Vernotte [29] postulated this equation independently. Eq. (1) can be used to represent the finite speed of heat and remove the infinite speed implication in the Fourier's law of heat conduction. Reviews of the use of Eq. (1) have been provided by Joseph and Preziosi [30,31] and Ozisik and Tzou [32], Tzou [33] has discussed the micro to macro scale behavior. Sharma [8–12] discussed the manifestation of the damped wave transport and relaxation equation in industrial applications and provided bounded solutions well within the constraints of second law of thermodynamics. Some investigators [34] have used the Boltzmann [35] transport equation and derived both the Fourier's law of heat conduction and the damped wave conduction and relaxation equation given by Eq. (1) as special cases. They derive a set of equations for length scales comparable to the mean free path of the molecule.

Tzou [36] attempted to provide the physical significance of the relaxation time in the wave theory of heat conduction. The relaxation time results from the phase-lag between the heat flux vector and the temperature gradient in a high-rate response. Brown and Churchil [37] measured the second sound. Peshkov [38] measured a thermal wave speed of 19 m/s at 1.4 K in Helium. Zehnder and Rosakis [39] measured the temperature distribution at the vicinity of the dynamically propagating cracks in 4340 steel. The relaxation mechanism is fundamental to thermal resonance that cannot be depicted by Fourier's law of heat conduction [40]. For the thermal wave speed around 900 m/s in 4340 steel at 480 °C the value of relaxation time was found to be of the order of 10^{-11} s. A table for relaxation time for materials is not available in the literature. Relaxation times for materials with a non-homogeneous inner structure were presented by Kaminski [41]. For Sodium bicarbonate they report a relaxation time of 29 s and 20 s for sand and 54 s for ion exchange materials. Mitura et al. [42] claim that the for the falling drying rate period the average relaxation time is of the order of several thousand seconds. For homogeneous substances the relaxation time values range from 10^{-8} to 10^{-10} s for gases, 10^{-10}–10^{-12} s for liquids and dielectric solids as concluded by Sieniutycz [43]. Mitra et al. [44] presented experimental evidence of the wave nature of heat propagation in processed meat and demonstrated that the hyperbolic heat conduction model is an accurate representation on a macroscopic level of the heat conduction process in such biological material. They report a relaxation time of the order of 16 s.

Some investigators have raised some concerns about the generalized Fourier's law of heat conduction violating the second law of thermodynamics [45–49] attempted to obtain analytical solution to the governing equations and found that the solution temperature for some values went above the boundary temperature indicating a possible violation of Clausius inequality. Transient instability, including the intrinsic transition from the desirable stability, neutral stability to the ultimate unstable response was investigated by Tzou [49] for a wide spectrum of heating rates. Tzou confirmed that the relaxation time results from the rate equation within the mainframe of the second law in the nonequilibrium, irreversible thermodynamics. Antaki [50] examined the dual phase-lag equation that was introduced by Tzou and provided an analytical solution for the case of a semi-infinite medium subject to constant wall flux boundary condition. Sharma [51] derived the damped wave conduction and relaxation equation from free electron theory and Stokes–Einstein formulation [52] and by analogy with mass diffusion. The relaxation time was found to be a third of the collision time of the electron and the obstacle. The velocity of heat was found to identical with the velocity of mass derived from kinetic representation of pressure or the Maxwell representation of the speed of molecules. A analytical solution for the case of finite slab subject to constant wall temperature was obtained. The final condition in time as the fourth condition for the second order hyperbolic PDE governing equation for the wave temperature was shown to result in well bounded solutions. This clearly means that care must go into the choice of the conditions used in the boundary of space and initial and final time domains. These have to be physically realistic such as at steady state equilibrium temperature is attained. Only for large relaxation times oscillations were found in the solution for temperature. These oscillations were subcritical and damped. The time conditions used by Taitel are unrealistic from the physical realities of heat transfer and that is why their solution exhibited a overshoot. Thus the equations do not violate the laws of thermodynamics as much as the choice of the space and time conditions as necessary constraints. Baumeister and Hamill [53] presented a analytical solution for the hyperbolic heat equation in a semi-infinite medium subject to a constant wall temperature boundary condition by the method of Laplace transforms. They found interface discontinuity in their solution. In this study the analytical solution of the damped wave conduction and relaxation equation under the constant wall flux boundary condition is examined for the case of semi-infinite medium in Cartesian coordinates using three different methods – (i) the method of Laplace transforms (ii) the generalized substitution and transformation of hyperbolic PDE into parabolic PDE and (iii) relativistic transformation of coordinates.

2. Method of laplace transforms

Consider the problem of one dimensional heat conduction and relaxation in a semi-infinite medium subject to a constant wall flux at one of the walls (Fig. 1). The semi-infinite medium possess constant thermophysical properties such as C_p, k, ρ, α and τ_r, i.e., heat capacity, thermal conductivity, density, thermal diffusivity and thermal relaxation time. Obtaining the dimensionless variables

$$u = \frac{(T - T_0)}{T_0}; \quad \tau = \frac{t}{\tau_r}; \quad X = \frac{x}{\sqrt{\alpha\tau_r}} \tag{2}$$

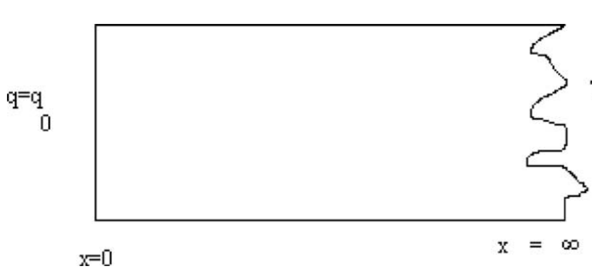

Fig. 1. Semi-infinite medium subject to constant wall flux at the surface.

The energy balance on a thin shell at x with thickness Δx is written. The governing equation can be obtained after eliminating q between the energy balance equation and the derivative with respect to x of the flux equation and introducing the dimensionless variables

$$\frac{\partial u}{\partial \tau} + \frac{\partial^2 u}{\partial \tau^2} = \frac{\partial^2 u}{\partial X^2} \tag{3}$$

The initial condition is

$$t = 0, \quad T = T_0; \quad u = 0$$

The second time condition can be assumed to come from zero accumulation at initial times and written as

$$t = 0, \quad \partial u/\partial \tau = 0$$

The boundary conditions are

$$X = \infty, \quad T = T_0; \quad u = 0 \tag{4}$$

$$X = 0, \quad q = q_o \tag{5}$$

Obtaining the Laplace transform of Eq. (2)

$$s\bar{u} - 0 + s^2\bar{u} - 0 - 0 = \frac{d^2\bar{u}}{dX^2} \tag{6}$$

It can be seen that the initial temperature, u and the accumulation term $\partial u/\partial \tau$ both are assumed zero at initial times.

Solving for the second order ordinary differential equation

$$u = a\exp(+X\sqrt{s(s+1)}) + b\exp(-X\sqrt{s(s+1)}) \tag{7}$$

From the boundary condition given by Eq. (4) as can be seen to be zero and from the boundary condition given by Eq. (5)

$$\frac{q_0^*}{s} = \frac{b\sqrt{s(s+1)}}{(s+1)} \text{orb} = \frac{q_0^*\sqrt{s+1}}{s^{3/2}} \tag{8}$$

$$u = \frac{q_0^*\sqrt{s+1}}{s^{3/2}}\exp(-X\sqrt{s(s+1)}) \tag{9}$$

Multiplying the numerator and denominator by sqrt$(s+1)$ and using the linear property of the Laplace transforms

$$\frac{u}{q_0^*} = \exp\left(\frac{-\tau}{2}\right)I_0(1/2\sqrt{\tau^2 - X^2}) + \int_0^\tau \exp\left(-\frac{\tau}{2}\right)I_0(1/2 \times \sqrt{p^2 - X^2})dp \tag{10}$$

The corresponding solution using the Fourier parabolic model can be given by [54]

$$\frac{u}{q_0^*} = \sqrt{4\tau}\int_x^\infty 1 - \text{erf}\left(\frac{x}{\sqrt{4\alpha t}}\right)d\left(\frac{x}{\sqrt{4\alpha t}}\right) \tag{11}$$

Thus the temperature profile in a semi-infinite medium subject to a constant wall flux is obtained. Eq. (10) is applicable in the open interval of $\tau > X$. In the open interval of $X > \tau$, it can be seen that the expression for dimensionless temperature can be written as

$$\frac{u}{q_0^*} = \exp\left(\frac{-\tau}{2}\right)J_0(1/2\sqrt{X^2 - \tau^2}) + \int_0^\tau \exp\left(-\frac{\tau}{2}\right)J_0(1/2 \times \sqrt{X^2 - p^2})dp \tag{12}$$

Eq. (12) can be arrived at by the realization that $I_0(i\eta) = J_0(\eta)$. The surface temperature is obtained by inversion of

$$\bar{u} = \frac{q_0^*}{s^{3/2}\sqrt{s+1}}$$
$$\frac{u}{q_0^*} = \exp\left(-\frac{\tau}{2}\right)I_0\left(\frac{\tau}{2}\right) + \tau\exp\left(-\frac{\tau}{2}\right)\left(I_0\left(\frac{\tau}{2}\right) + I_1\left(\frac{\tau}{2}\right)\right) \tag{13}$$

The flux expression in the Laplace domain can be obtained by noting that $q^*(s+1) = -\mathrm{d}u/\mathrm{d}X$ and by differentiating Eq. (9) with respect to X and seen to be

$$\bar{q}^* = \frac{q_0^*}{s}\exp(-X\sqrt{s(s+1)}) \tag{14}$$

Eq. (14) can also be obtained by writing the governing equation in the semi-infinite medium in one dimension in terms of heat flux instead of in terms of temperature as written in Eq. (3). The governing equation in terms of dimensionless heat flux can be seen to be

$$\frac{\partial q^*}{\partial \tau} + \frac{\partial q^*}{\partial \tau^2} = \frac{\partial q^*}{\partial X^2} \tag{15}$$

The Laplace transform of Eq. (14) upon substitution of boundary conditions at $X = \infty$ and $X = 0$ can be seen to yield Eq. (14). The inversion of this expression given in Eq. (13) can be obtained as

$$\frac{q}{q_0} = \exp\left(-\frac{X}{2}\right) + X\int_X^\tau \exp\left(-\frac{p}{2}\right)\frac{I_1(1/2\sqrt{p^2 - X^2})}{\sqrt{p^2 - X^2}}\mathrm{d}p \tag{16}$$

It can be realized that Eq. (16) is applicable in the open interval, $\tau > X$. At the wave front, the solution for dimensionless heat flux can be obtained as follows. The damping decaying exponential in time can be divided out by supposing that the wave heat flux, W is such that $u = W\exp(-\tau/2)$. It can be seen that this substitution transforms Eq. (15) into

$$\frac{\partial^2 W}{\partial \tau^2} - \frac{W}{4} = \frac{\partial^2 W}{\partial X^2} \tag{17}$$

Let $\eta = \tau^2 - X^2$. It can be seen that Eq. (17) becomes

$$4\eta\frac{\partial W}{\partial \eta} + 4\frac{\partial W}{\partial \eta} - \frac{W}{4} = 0 \tag{18}$$

At the wave front, $\eta = 0$ and Eq. (18) becomes,

$$\frac{\partial W}{\partial \eta} = \frac{W}{16} \tag{19}$$

The solution of Eq. (19) can be seen to be $W = c\exp(\eta/16)$ or $W = c$ at the wave front. Thus at the wave front, $u = c\exp(-\tau/2) = c\exp(-X/2)$. From the boundary condition given by Eq. (5) c can be seen to be 1. At the wave front

$$q^* = \exp\left(-\frac{X}{2}\right) = \exp\left(-\frac{\tau}{2}\right) \tag{20}$$

The dimensionless temperature can be obtained from the energy balance equation, i.e., $\frac{\partial q^*}{\partial X} = -\frac{\partial u}{\partial \tau}$. The dimensionless temperature can be seen to at the wave front

$$u = 1 - \exp\left(-\frac{\tau}{2}\right) \tag{21}$$

3. A generalized substitution for the generalized Fourier's law of heat conduction

The governing equation for transient heat flux in one dimension in Cartesian coordinates can be given by

$$\frac{\partial^2 q^*}{\partial \tau^2} + \frac{\partial q^*}{\partial \tau} = \frac{\partial^2 q^*}{\partial X^2} \tag{22}$$

The parabolic PDE that this can become after a suitable transformation is

$$\frac{\partial q^*}{\partial \tau} = \beta\frac{\partial^2 q^*}{\partial \theta^2} \tag{23}$$

Let θ be a general substitution

$$\theta = g(X) + V(\tau) \tag{24}$$

Then

$$\frac{\partial q^*}{\partial \tau} = V'(\tau)\frac{\partial u}{\partial \theta} \tag{25}$$

$$\frac{\partial^2 q^*}{\partial \tau^2} = V'^2\frac{\partial^2 q^*}{\partial \theta^2} + V''\frac{\partial q^*}{\partial \theta} \tag{26}$$

$$\frac{\partial^2 q^*}{\partial X^2} = g'^2\frac{\partial^2 q^*}{\partial \theta^2} + g''\frac{\partial q}{\partial \theta} \tag{27}$$

Substituting Eqs. (26), (27) in Eq. (25)

$$\frac{\partial q^*}{\partial \tau} = \frac{\partial^2 q^*}{\partial X^2} - \frac{\partial^2 q^*}{\partial \tau^2} = (g'^2 - V'^2)\frac{\partial^2 q^*}{\partial \theta^2} + (g'' - V')\frac{\partial q^*}{\partial \theta} \tag{28}$$

Inorder for Eq. (28) to take on the form of Eq. (23)

$$g'' - V'' = 0 \tag{29}$$

$$g'^2 - V'^2 = \beta \tag{30}$$

Thus g' = sqrt$(1 + V'2) = c$ (only then two different functions can be equal)

$$g = cZ; g'' = 0 \quad \text{and} \quad V'' = 0 \tag{31}$$

$$V' = d; \quad V = \mathrm{d}\tau + e \tag{32}$$

$$c^2 - d^2 = \beta \tag{33}$$

$$d = \mathrm{sqrt}(c^2 - \beta) \tag{34}$$

Hence,

$$g'^2 - V'^2 = \beta \tag{35}$$

$$\frac{\partial q^*}{\partial \tau} = \beta\frac{\partial^2 q^*}{\partial \theta^2} \tag{36}$$

The substitution, θ that made the transformation possible was,

$$\theta = c\left(X + \tau\sqrt{1 - \frac{\beta}{c^2}}\right) + e \tag{37}$$

Assuming $c = 1$ and $e = 0$

$$\theta = X \pm \tau\sqrt{1-\beta} \tag{38}$$

Eq. (36) is a PDE in two variables. One of the variables is the time variable and the other is the lumped variable that is a function of both space and time. This has spatiotemporal symmetry. This can be converted into a ODE in one variable by the Boltzmann transformation.

Let $\psi = \frac{\theta}{\sqrt{4\beta\tau}}$

$$\beta\frac{\partial^2 q^*}{\partial \theta^2} = \frac{1}{4\tau}\frac{\partial^2 q^*}{\partial \psi^2} \tag{39}$$

$$\frac{\partial q^*}{\partial \tau} = \frac{-\psi}{4\tau}\frac{\partial q^*}{\partial \psi} \tag{40}$$

Substituting Eqs. (39), (40) into Eq. (36)

$$\frac{\partial^2 q^*}{\partial \psi^2} = -2\psi\frac{\partial q^*}{\partial \psi} \tag{41}$$

Let, $p = \frac{\partial q^*}{\partial \psi}$

$$-\frac{\partial p}{p} = 2\psi\partial\psi \tag{42}$$

Integrating both sides

$$\ln(p) = -\psi^2 + c' \tag{43}$$

$$q = \int \exp(-\psi^2)\mathrm{d}\psi + c''$$

$$\mathrm{erf}(\psi) = \frac{2}{\sqrt{\pi}} \int_0^{\psi} \exp(-\psi^2)\mathrm{d}\psi \tag{44}$$

Thus

$$q = c_1 \mathrm{erf}(\psi) + c_2 \tag{45}$$

From the boundary condition at $\psi = \infty$, where $\mathrm{erf}(\infty) = 1$, $q = 0$.

$$c_1 = -c_2 \tag{46}$$

From the boundary condition at the surface, $q = q_o$, when $X = 0$

$$\theta = \tau\sqrt{1-\beta}; \quad \psi = \frac{\sqrt{\tau(1-\beta)}}{2\sqrt{\beta}} \tag{47}$$

$$q_0 = c_2\left(1 - \frac{\mathrm{erf}(\sqrt{\tau(1-\beta)})}{2\sqrt{\beta}}\right) \tag{48}$$

$$q = c_2(1 - \mathrm{erf}(\psi)) \tag{49}$$

Eliminating c_2 between the two Eqs. (48) and (49)

$$q^* = \frac{1 - \mathrm{erf}(\psi)}{\left(1 - \frac{\mathrm{erf}(\sqrt{\tau(1-\beta)})}{2\sqrt{\beta}}\right)} \tag{50}$$

β can be chosen as follows

$$1 = \frac{\sqrt{1-\beta}}{2\sqrt{\beta}} \tag{51}$$

$$4\beta + \beta - 1 = 0 \tag{52}$$

$$\beta = 1/5 \tag{53}$$

Then

$$q^* = \left[\frac{1 - \mathrm{erf}\left(\frac{\sqrt{5}X}{\sqrt{4\tau}} + \sqrt{\tau}\right)}{1 - \mathrm{erf}\sqrt{\tau}}\right] \tag{54}$$

The parabolic PDE can be described by two space and one time condition. For a PDE of order n, n functions need to be solved for as against n constants for a ODE of order n. Although the hyperbolic PDE needs two space and two time conditions for complete description, it was converted to a parabolic PDE.

4. Method of relativistic transformation of coordinates

The energy balance on a thin spherical shell at x with thickness Δx is written in one dimension is written as $-\partial q/\partial x = \rho C_p \partial T/\partial t$. The governing equation can be obtained in terms of the flux after eliminating the temperature between the energy balance equation and the non-Fourier expression. This is achieved by differentiating the generalized Fourier's law of heat conduction wrt to time and the energy balance equation wrt to x and then eliminating the second cross derivative of the temperature with respect to space and time

$$\frac{\partial q^*}{\partial \tau} + \frac{\partial^2 q^*}{\partial \tau^2} = \frac{\partial^2 q^*}{\partial X^2} \tag{55}$$

It can be seen the governing equation for the dimensionless heat flux is identical in form with that of the dimensionless temperature. The initial condition is

$$\tau = 0, \quad q^* = 0 \tag{56}$$

The boundary conditions are

$$X = \infty, \quad q^* = 0 \tag{57}$$

$$X = 0, \quad q^* = 1 \tag{58}$$

Let us suppose that the solution for q^* is of the form $W\exp(-n\tau)$ for $\tau > 0$ where W is the transient wave flux. Then, For $n = 1/2$ Eq. (48) becomes

$$\frac{\partial^2 W}{\partial \tau^2} - \frac{W}{4} = \frac{\partial^2 W}{\partial X^2} \tag{59}$$

The solution to Eq. (59) can be obtained by the following relativistic transformation for $\tau > X$.

Let $\eta = (\tau^2 - X^2)$

$$\frac{\partial^2 W}{\partial \tau^2} = 4\tau^2 \frac{\partial^2 W}{\partial \eta^2} + 2\frac{\partial W}{\partial \eta} \tag{60}$$

$$\frac{\partial^2 W}{\partial X^2} = 4X^2 \frac{\partial^2 W}{\partial \eta^2} + 2\frac{\partial W}{\partial \eta} \tag{61}$$

Combining Eqs. (60), (61) into Eq. (59),

$$4(\tau^2 - X^2)\frac{\partial^2 W}{\partial \eta^2} + 4\frac{\partial W}{\partial \eta} - \frac{W}{4} = 0 \tag{62}$$

$$\eta^2 \frac{\partial^2 W}{\partial \eta^2} + \eta \frac{\partial W}{\partial \eta} - \frac{\eta W}{16} = 0 \tag{63}$$

Eq. (63) can be seen to be a special differential equation in one independent variable. The number of variables in the hyperbolic PDE has thus been reduced from two to one. Comparing Eq. (56) with the generalized form of Bessel's equation it can be seen that $a = 1$, $b = 0$, $c = 0$, $s = 1/2$, $d = -1/16$. The order of the solution is calculated as 0 and the general solution is given by

$$W = c_1 I_0\left(\frac{\sqrt{\eta}}{2}\right) + c_2 K_0\left(\frac{\sqrt{\eta}}{2}\right) \tag{64}$$

The wave flux W, is finite when $\eta = 0$ and hence it can be seen that c_2 can be seen to be zero. The c_1 can be solved from the boundary condition given in Eq. (51). The expression for the dimensionless heat flux for times τ, greater than X is thus,

$$q^* = c_1 \exp\left(-\frac{\tau}{2}\right) I_0\left(\frac{\sqrt{\tau^2 - X^2}}{2}\right) \tag{65}$$

From the boundary condition at the surface

$$q^* = 1 = c_1 \exp\left(-\frac{\tau}{2}\right) I_0\left(\frac{\sqrt{\tau^2 - X^2}}{2}\right) \tag{66}$$

Eliminating c_1 between Eqs. (58) and (59) a approximate solution for the heat flux can be obtained as

$$q^* = \frac{I_0\left(\frac{\sqrt{\tau^2 - X^2}}{2}\right)}{I_0\left(\frac{\tau}{2}\right)} \tag{67}$$

Eq. (60) is applicable in the open interval, $\tau > X$. At the wave front it can be seen that Eq. (55) can be solved and the dimensionless heat flux found as $q^* = \exp(-X/2) = \exp(-\tau/2)$. In the open interval $X > \tau$, the expression for heat flux q^* can be seen to be

$$q^* = \frac{J_0\left(\frac{\sqrt{\tau^2 - X^2}}{2}\right)}{I_0\left(\frac{\tau}{2}\right)} \tag{68}$$

This can be done in one of two ways. $I_0(i\eta) = J_0(\eta)$. Substituting this in Eq. (67) in the open interval, $X > \tau$, Eq. (67) becomes Eq. (68). The second way is to redefine $\eta = X^2 - \tau^2$. Eq. (63) then becomes

$$\eta^2 \frac{\partial^2 W}{\partial \eta^2} + \eta \frac{\partial W}{\partial \eta} + \frac{\eta W}{16} = 0 \tag{69}$$

Comparing Eq. (69) with the generalized Bessel differential equation and solving for the integration variables given the boundary conditions given by Eq. (58), Eq. (68) results. It can be seen that $J_0(i\tau/2) = I_0(\tau/2)$. From Eq. (69) the inertial lag time associated with a interior point in the semi-infinite medium can be calculated by realizing that the first zero of the Bessel function, $J_0(\psi)$ occurs at $\psi = 2.4048$. Thus,

$$2.4048^2 = \frac{x_p^2}{\alpha\tau_r} - \frac{t_{lag}^2}{\tau_r^2} \tag{70}$$

$$t_{lag} = \sqrt{x_p^2\frac{\tau_r}{\alpha} - 5.7831\tau_r^2} \tag{71}$$

The penetration distance for a given time instant can be developed at the first zero of the Bessel function. Beyond this point by the interior temperatures can be no less than the initial temperature. Thus,

$$X_{pen} = \sqrt{5.7831 + \tau^2} \tag{72}$$

5. Discussion

In order to further study the dimensionless heat flux from the hyperbolic damped wave conduction and relaxation equation, the integral expression given in Eq. (16) can be simplified using a Chebyshev polynomial [55]. Chebyshev polynomial approximations tend to distribute the errors more evenly with reduced maximum error by use of cosine functions. The set of polynomials, $T_n(r) = \cos(n\theta)$ generated from the sequence of cosine functions using the transformation

$$\theta = \cos^{-1}(r) \tag{73}$$

is called Chebyshev polynomials (Table 1). Coefficients of the Chebyshev polynomials for the integrand in Eq. (16), $\frac{I_1 1/2\sqrt{p^2-X^2}}{\sqrt{p^2-X^2}}$ can be computed with some effort. The modified Bessel function of the first order and first kind can be expressed as a power series as follows

$$\frac{I_1 1/2\sqrt{p^2-X^2}}{\sqrt{p^2-X^2}} = \sum_{m=0}^{\infty}\frac{(p^2-X^2)^m}{4^{2k+1}(m!)(m+1)!} = \frac{\psi^m}{4^{2k+1}(m!)(m+1)!} \tag{74}$$

where $\psi = p^2 - X^2$.

Each of the ψ^m term can be replaced with its expansion in terms of Chebyshev polynomials given in Table 2.

The coefficients of like polynomials $T_i(r)$ are collected. When the truncated power series polynomial of the integrand Eq. (71) is represented by Chebyshev polynomial, some of the high-order Chebyshev polynomials can be dropped with negligible truncation error. This is because the upper bound for $T_n(r)$ in the interval $(-1, 1)$ is 1. The truncated series can then be re-transformed to a polynomial in r with fewer terms than the original and with modified coefficients. This procedure is referred to as Chebyshev economization or telescoping a power series.

Prior to expression of Eq. (74) in terms of Chebyshev polynomials the interval (X, τ) needs to be converted to the interval $(-1, 1)$. So let

Table 1
Chebyshev polynomials

$T_0(r) = 1$
$T_1(r) = r$
$T_2(r) = 2r^2 - 1$
$T_3(r) = 4r^3 - 3r$
$T_4(r) = 8r^4 - 8r^2 + 1$
$T_5(r) = 16r^5 - 20r^3 + 5r$
$T_6(r) = 32r^6 - 48r^4 + 18r^2 - 1$

Table 2
Powers of r in terms of the Chebyshev polynomials

$1 = T_0(r)$
$r = T_1(r)$
$r^2 = \frac{1}{2}(T_0(r) + T_2(r))$
$r^3 = \frac{1}{4}(3T_1(r) + T_3(r))$
$r^4 = \frac{1}{8}(3T_0(r) + 4T_2(r) + T_4(r))$
$r^5 = \frac{1}{16}(10T_1(r) + 5T_3(r) + T_5(r))$
$r^6 = \frac{1}{32}(10T_0(r) + 15T_2(r) + 6T_4(r) + T_6(r))$

$$r = \frac{2\psi - \tau - X}{\tau - X} \quad \text{and} \quad \psi = \frac{r(\tau - X) + (\tau + X)}{2} \tag{75}$$

Further let

$$\xi = (\tau - X) \quad \text{and} \quad \eta = (\tau + X) \tag{76}$$

Thus,

$$\psi = \frac{r\xi + \eta}{2} \tag{77}$$

Substituting Eq. (77) in Eq. (74),

$$\frac{I_1 1/2(p^2-X^2)}{\sqrt{p^2-X^2}} = \sum_{m=0}^{\infty}\frac{(r\xi+\eta)^m}{2^k 4^{2k+1} m!(m+1)!} \tag{78}$$

The RHS (right hand side) of Eq. (78) can be written as

$$\text{RHSEq. (78)} = \frac{1}{4} + \frac{r\xi+\eta}{256} + \frac{(r\xi+\eta)^2}{49,152} + \cdots \tag{79}$$

A truncation error of $\frac{(r\xi+\eta)^3}{18,874,368}$ is incurred in writing the LHS of Eq. (78) as Eq. (79). Replacing the r, r^2, r^3 terms in Eq. (78) in terms of Chebyshev polynomials given in Table 1 and collecting the like Chebyshev coefficients, T_0, T_1 and T_2, the RHS of Eq. (75) can be written as

$$T_0(r)\left(\frac{1}{4} + \frac{\eta}{256} + \frac{\eta^2}{49,152} + \cdots + \frac{\xi^2}{98,304}\right) + T_1(r)\left(\frac{\xi}{256} + \frac{2\eta\xi}{49,152} + \cdots\right) \tag{80}$$

The $T_2(r)$ term can be dropped with an added error of only $\frac{\xi^2}{98,304}$. The order of magnitude of the error incurred is thus, $O\left(\frac{\xi^2}{98,304}\right)$. Retransformation of the series given by Eq. (77) yields

$$\frac{I_1 1/2\sqrt{p^2-X^2}}{\sqrt{p^2-X^2}} = \frac{1}{4} - \frac{X^2}{128} + \frac{\eta^2}{49,152} + \frac{\xi^2}{98,304} + \frac{(p^2-X^2)}{128} \tag{81}$$

The error involved in writing Eq. (81) is $4.1 \times 10^{-5}\eta\xi$. If Chebyshev polynomial approximation was not used for the integrand and the power series was truncated after the second term, the error would have been, $4 \times 10^{-3}r^2$. Substituting Eq. (81) in Eq. (16) and further integrating the expression for dimensionless heat flux

$$u = \exp\left(-\frac{X}{2}\right) + X\exp\left(-\frac{X}{2}\right)\left(\frac{5}{8} + \frac{X}{16} + \frac{\eta^2}{24,576} + \frac{\xi^2}{49.152}\right) + X\exp\left(-\frac{\tau}{2}\right)\left(\frac{3}{8} - \frac{\tau}{16} - \frac{X^2}{64} + \frac{\eta^2}{24,576} + \frac{\xi^2}{49,152}\right) \tag{82}$$

It can be seen that Eq. (82) can be expected to yield reliable predictions on the transient temperature close to the wave front. This is because the error increases as a function of $4.1 \times 10^{-5}\xi\eta$. Far from the wave front, i.e, close to the surface the numerical error may become significant.

The dimensionless heat flux solution obtained after the Chebyshev polynomial approximation for the integrand in Eq. (16) and further integration is shown in Fig. 2. The conditions selected was for typical $\tau = 8$ Eq. (82) was plotted using a MS Excel spreadsheet. This is shown in Fig. 2. The expression for temperature developed by using the method of relativistic transformation for the same condition of $\tau = 8$ is also shown side by side in Fig. 2. It can be seen that both the Laplace transform solution and the solution from the relativistic transformation are close to each other, within a average of 12% deviation from each other. It can also be seen that close to the surface or far from the wavefront the numerical errors expected from the Chebyshev polynomial approximation is large. For such conditions the expression developed by the method of relativistic transformation may be used. For conditions close to the wavefront the further integrated expression developed in this study may be used. The penetration dimensionless distance for $\tau = 8$ beyond which there is expected no heat transfer is given by Eq. (34) and is 8.35 by the method of relativistic transformation. The Laplace transform solution solution is only for $\tau > X$. Both the solutions for transient temperature for the damped wave conduction and relaxation hyperbolic equation from the method of Laplace transforms and Chebyshev economization and the method of relativistic transformation are compared against the prediction for transient temperature by the Fourier parabolic heat conduction model. The transient temperature from the Chebyshev economization found to be within 25% of the error function solution for the parabolic Fourier heat conduction model. The hyperbolic model solutions compare well with the Fourier model solution for transient temperature close to the wave front and close to the surface (to within 15% of each other). The deviations are at the intermediate values. As can be seen from Fig. 2, the integrated expression appears to reach the boundary temperature at $X = 1.3$. It cannot be concluded whether this is within numerical error ir whether it is a violation of second law of thermodynamics. Further work is needed to clarify this point.

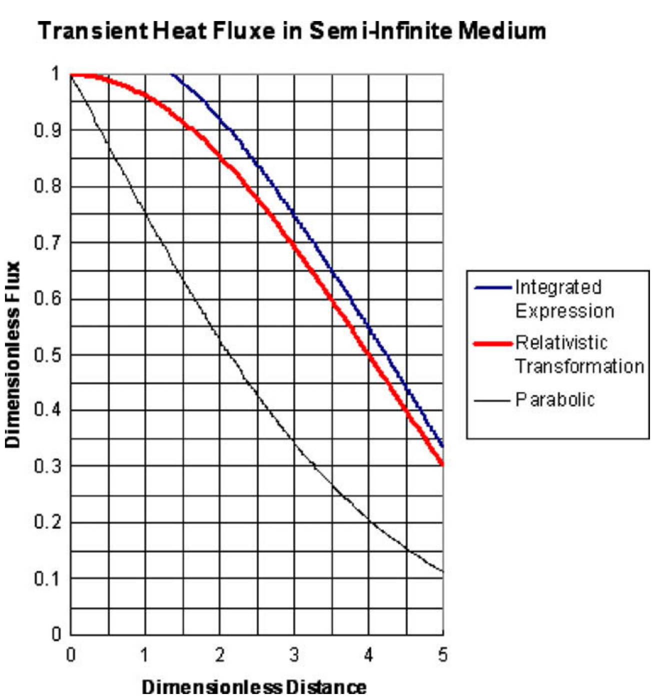

Fig. 2. Dimensionless heat flux in semi-infinite medium during damped wave conduction and relaxation, $\tau = 8$ and parabolic Fourier heat conduction.

6. Conclusions

Bounded solutions were obtained for the damped wave conduction and relaxation equation in one dimension in Cartesian coordinates for a semi-infinite medium subject to the constant wall flux boundary condition. Three different methods were employed. In the first approach the method of Laplace transforms was used. The constant wall flux case lends itself to a Laplace transform expression whose inversion is readily available by looking up the tables for the temperature profile. The solutions for both the temperature and flux are provided. The solutions are domain restricted. Three regimes can be identified (a) zero transferring regime; (b) rising regime and (c) falling regime. In the second approach a generalized substitution is examined to convert the hyperbolic PDE into a parabolic PDE. The transform selected is one with spatiotemporal symmetry. The resulting parabolic PDE can be solved for using the Boltzmann transformation. Assumptions were made for the lumped ordinate where needed. The solution is simpler in form compared with those reported in the literature for similar systems. It is not clear what this means in terms of fundamental mechanisms for heat conduction. In the third approach the damping term was first removed from the governing equation. The resulting equation was transformed into a Bessel differential equation using a spatiotemporal symmetric transformation variable. A approximate solution for the flux was obtained. The inertial regime, rising and falling regimes were identified in the solution. The solution for dimensionless heat flux is continuous at the wave front. The heat flux and temperature at the wave front regardless of the method can be given by Eqs. (20) and (21), respectively. For short times the removal of damping term from the hyperbolic PDE may be a reasonable assumption as $u = W\exp(-\tau/2)$. For short times, the transient temperature is the wave temperature. For considerable times the wave temperature decays out and becomes monotonic as described by the Fourier parabolic model. The exact solution for the hyperbolic PDE consists of three regimes – a inertial regime, a rising regime characterized by Bessel composite function in space and time and a third regime characterized by a modified Bessel composite function in space and time.

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